

# BRST symmetries in free particle system on toric geometry

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We study a free particle system residing on a torus to investigate its Becci-Rouet-Stora-Tyutin symmetries associated with its Stückelberg coordinates, ghosts and anti-ghosts. By exploiting zeibein frame on the toric geometry, we evaluate energy spectrum of the system to describe the particle dynamics. We also investigate symplectic structures involved in the second-class system on the torus.

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## I. INTRODUCTION

It is well known in string theory that toric geometry is a generalization of the projective identification that defines  $CP^n$  corresponding to the most general linear sigma model, and it provides a scheme for constructing Calabi-Yau manifolds and their mirrors [1]. Recently, on the basis of boundary string field theory [2], the brane-antibrane system was exploited [3] in the toroidal background to investigate its thermodynamic properties associated with the Hagedorn temperature [4, 5]. The Nahm transform and moduli spaces of  $CP^n$  models were also studied on the toric geometry [6]. In a four dimensional, toroidally compactified heterotic string, the electrically charged BPS-saturated states were shown to become massless along the hypersurfaces of enhanced gauge symmetry of a two-torus moduli subspace [7].

The Becci-Rouet-Stora-Tyutin (BRST) symmetries [8] have been constructed for constrained systems [9] in the Batalin-Fradkin-Vilkovisky (BFV) scheme [10]. In order to treat rigorously the constraints involved in the systems, the Dirac quantization was proposed [11], and it was later improved by converting a second-class constraint system into a first-class one in the Batalin-Fradkin-Tyutin (BFT) embedding [12], where the BRST symmetries can be generated.

Recently, the gauge symmetry enhancement was studied [13] on target space of Grassmann manifold in the Dirac Hamiltonian formalism, and the BRST invariant effective Lagrangian was also realized in noncommutative D-brane system with NS  $B$ -field [14]. To show novel phenomenological aspects [15], the compact form of the first-class Hamiltonian has been also constructed [16] for the  $O(3)$  nonlinear sigma model, which has been also studied [17] to investigate the symplectic structures [18] and BFT embeddings.

In this paper, we will construct a first-class Hamiltonian by introducing the Stückelberg coordinates associated with the geometrical constraints on the torus. In the BFV scheme, we will then find a BRST-invariant gauge fixed Lagrangian including ghosts and anti-ghosts, and the corresponding BRST transformation rules. We will also construct the spectrum and the symplectic structures of the free particle system on the torus. In Sec. II, we will introduce a free particle system residing on a torus, to construct the first-class Hamiltonian and Dirac brackets. In Sec. III, we will introduce canonical sets of ghosts and anti-ghosts in the BFV scheme, to construct the BRST symmetric effective Lagrangian. In Sec. IV, we will obtain the spectrum of the free particle on the torus to figure out the particle dynamics. In Sec. V, we will study the symplectic structures involved in the second-class system on the torus.

## II. FIRST-CLASS CONSTRAINTS AND HAMILTONIAN

In this section, we consider a free particle system residing on a torus, whose Lagrangian is of the form

$$L_0 = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m(b+r\sin\theta)^2\dot{\phi}^2. \quad (2.1)$$

where we have used toroidal coordinates  $(r, \theta, \phi)$  for toric geometry

$$x_1 = (b+r\sin\theta)\cos\phi, \quad x_2 = (b+r\sin\theta)\sin\phi, \quad x_3 = r\cos\theta, \quad (2.2)$$

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to satisfy

$$((x_1^2 + x_2^2)^{1/2} - b)^2 + x_3^2 = r^2. \quad (2.3)$$

Note that we have now a torus with axial circle in the  $x_1$ - $x_2$  plane centered at the origin, of radius  $b$ , having a circular cross section of radius  $r$ , and the angle  $\theta$  ranges from 0 to  $2\pi$ , and the angle  $\phi$  from 0 to  $2\pi$ . To fulfill the toric geometry (2.3), we can also exploit another toroidal coordinates  $(\mu, \eta, \phi)$  defined as [19]

$$x_1 = \frac{c \sinh \mu \cos \phi}{\cosh \mu - \cos \eta}, \quad x_2 = \frac{c \sinh \mu \sin \phi}{\cosh \mu - \cos \eta}, \quad x_3 = \frac{c \sin \eta}{\cosh \mu - \cos \eta}, \quad (2.4)$$

where  $\mu$  ranges from 0 to  $\infty$ ,  $\eta$  from 0 to  $2\pi$ , and  $\phi$  from 0 to  $2\pi$ . Here, we have relations between these two coordinate systems (2.2) and (2.4),

$$r = \frac{c}{\sinh \mu}, \quad \theta = \cos^{-1} \frac{\sinh \mu \sin \eta}{\cosh \mu - \cos \eta}, \quad \phi = \phi, \quad b = c \coth \mu. \quad (2.5)$$

Now, we impose the condition that the particle is constrained to satisfy a geometrical constraint

$$\Omega_1 = r - a \approx 0. \quad (2.6)$$

By performing the Legendre transformation, one can obtain the canonical Hamiltonian,<sup>1</sup>

$$H_0 = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2m(b + r \sin \theta)^2} \quad (2.7)$$

where  $p_r$ ,  $p_\theta$  and  $p_\phi$  are the canonical momenta conjugate to the coordinates  $r$ ,  $\theta$  and  $\phi$ , respectively, given by

$$p_r = m\dot{r}, \quad p_\theta = mr^2\dot{\theta}, \quad p_\phi = m(b + r \sin \theta)^2\dot{\phi}. \quad (2.8)$$

The time evolution of the constraint  $\Omega_1$  yields an additional secondary constraint

$$\Omega_2 = p_r \approx 0 \quad (2.9)$$

and  $\Omega_1$  and  $\Omega_2$  form a second-class constraint algebra

$$\Delta_{kk'} = \{\Omega_k, \Omega_{k'}\} = \epsilon_{kk'} \quad (2.10)$$

with  $\epsilon_{12} = -\epsilon_{21} = 1$ . Since the constraints are second-class, we can define Dirac bracket

$$\{F, G\}_D = \{F, G\} - \{F, \Omega_k\}\Delta^{kk'}\{\Omega_{k'}, G\} \quad (2.11)$$

with  $\Delta^{kk'}$  being the inverse of  $\Delta_{kk'}$  in (2.10) to yield

$$\{r, p_r\}_D = 0, \quad \{\theta, p_\theta\}_D = 1, \quad \{\phi, p_\phi\}_D = 1. \quad (2.12)$$

In the forthcoming section, the Dirac brackets (2.12) will be discussed in terms of symplectic brackets. In the quantum level, these Dirac brackets produce the following commutator relations

$$[r, p_r] = 0, \quad [\theta, p_\theta] = i\hbar, \quad [\phi, p_\phi] = i\hbar. \quad (2.13)$$

Following the BFT embedding [12] which systematically converts the second-class constraints into first-class ones, we introduce the Stückelberg coordinates  $(\eta, p_\eta)$  with the Poisson brackets

$$\{\eta, p_\eta\} = 1,$$

<sup>1</sup> One can include the constraint (2.6) explicitly in the Lagrangian to yield  $L = L_0 + u(r - a)$  with a Lagrangian multiplier  $u$ . One can then obtain a primary constraint  $\Omega_0 = p_u$  with  $p_u$  being momentum conjugate to  $u$ . The Hamiltonian is then given by  $H_T = H_0 - u(r - a)$  and successive time evolutions of  $\Omega_0$  reproduce  $\Omega_1 = r - a$  and  $\Omega_2 = p_r$ . The condition  $\{\Omega_2, H_T\} = 0$  fixes value of  $u$ , namely  $u = -p_\theta^2/(mr^3) - p_\phi^2 \sin \theta/(m(b + r \sin \theta)^3)$ , which can terminate series of constraints. Since  $\Omega_0$  is first-class, one can thus end up with two second-class constraints  $\Omega_1$  and  $\Omega_2$ , which are used in the context.

to obtain the first-class constraints as follows

$$\begin{aligned}\tilde{\Omega}_1 &= \Omega_1 - \eta = r - a - \eta, \\ \tilde{\Omega}_2 &= \Omega_2 + p_\eta = p_r + p_\eta.\end{aligned}\tag{2.14}$$

Note that these first-class constraints yield a strongly involutive first-class constraint algebra  $\{\tilde{\Omega}_i, \tilde{\Omega}_j\} = 0$ , which is related with the first Dirac bracket in (2.12) and that the particle is geometrically constrained to reside on the torus with the modified radius  $r = a + \eta$  in the extended phase space.

Next, we construct the first-class Hamiltonian  $\tilde{H}_0$  as a power series in the Stückelberg coordinates  $(\eta, p_\eta)$  by demanding that they are strongly involutive:  $\{\tilde{\Omega}_i, \tilde{H}_0\} = 0$ . After some algebra, we obtain the first-class Hamiltonian,

$$\tilde{H}_0 = \frac{(p_r + p_\eta)^2}{2m} + \frac{p_\theta^2}{2m(r - \eta)^2} + \frac{p_\phi^2}{2m(b + (r - \eta)\sin\theta)^2}.\tag{2.15}$$

A problem with  $\tilde{H}_0$  in (2.15) is that it does not naturally generate the first-class Gauss law constraint from the time evolution of the constraint  $\tilde{\Omega}_1$ . By introducing an additional term proportional to the first-class constraints  $\tilde{\Omega}_2$  into  $\tilde{H}_0$ , we obtain an equivalent first-class Hamiltonian

$$\tilde{H} = \tilde{H}_0 - p_\eta \tilde{\Omega}_2,\tag{2.16}$$

which naturally generates the Gauss law constraint

$$\{\tilde{\Omega}_1, \tilde{H}\} = \tilde{\Omega}_2, \quad \{\tilde{\Omega}_2, \tilde{H}\} = 0.\tag{2.17}$$

One notes here that  $\tilde{H}_0$  and  $\tilde{H}$  act in the same way on physical states, which are annihilated by the first-class constraints. Similarly, the equations of motion for observables remain unaffected by the additional term in  $\tilde{H}$ . Furthermore, in the limit  $(\eta, p_\eta) \rightarrow 0$ , our first-class system is exactly reduced to the original second-class one.

### III. BRST SYMMETRIES AND EFFECTIVE LAGRANGIAN

In this section, we introduce canonical sets of the ghosts and anti-ghosts together with the Lagrangian multipliers in the BFV scheme [10], which is applicable to theories with the first-class constraints,

$$(\mathcal{C}^i, \bar{\mathcal{P}}_i), \quad (\mathcal{P}^i, \bar{\mathcal{C}}_i), \quad (N^i, B_i), \quad (i = 1, 2),$$

which satisfy the super-Poisson algebra

$$\{\mathcal{C}^i, \bar{\mathcal{P}}_j\} = \{\mathcal{P}^i, \bar{\mathcal{C}}_j\} = \{N^i, B_j\} = \delta_j^i.$$

Here the super-Poisson bracket is defined as

$$\{A, B\} = \frac{\delta A}{\delta q}|_r \frac{\delta B}{\delta p}|_l - (-1)^{\eta_A \eta_B} \frac{\delta B}{\delta q}|_r \frac{\delta A}{\delta p}|_l,$$

where  $\eta_A$  denotes the number of fermions, called the ghost number, in  $A$  and the subscript  $r$  and  $l$  denote right and left derivatives, respectively.

In our model, the nilpotent BRST charge  $Q$  defined as

$$Q = \mathcal{C}^i \tilde{\Omega}_i + \mathcal{P}^i B_i,\tag{3.1}$$

and the BRST invariant minimal Hamiltonian  $H_m$  given by

$$H_m = \tilde{H} - \mathcal{C}^1 \bar{\mathcal{P}}_2,\tag{3.2}$$

satisfy the relations

$$\{Q, H_m\} = 0, \quad Q^2 = \{Q, Q\} = 0.\tag{3.3}$$

Our next task is to fix the gauge, which is crucial to identify the BFT auxiliary coordinate  $\eta$  with the Stückelberg coordinate. The desired identification follows if we choose the fermionic gauge fixing function  $\Psi$  as

$$\Psi = \bar{\mathcal{C}}_i \chi^i + \bar{\mathcal{P}}_i N^i,\tag{3.4}$$

with the unitary gauge

$$\chi^1 = \Omega_1, \quad \chi^2 = \Omega_2. \quad (3.5)$$

Here note that the  $\Psi$  satisfies the following identity

$$\{\{\Psi, Q\}, Q\} = 0. \quad (3.6)$$

The effective quantum Lagrangian is then described as

$$L_{eff} = p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} + p_\eta \dot{\eta} + B_2 \dot{N}^2 + \bar{\mathcal{P}}_i \dot{\mathcal{C}}^i + \bar{\mathcal{C}}_2 \dot{\mathcal{P}}^2 - H_{tot} \quad (3.7)$$

where  $H_{tot} = H_m - \{Q, \Psi\}$  and the terms  $B_1 \dot{N}^1 + \bar{\mathcal{C}}_1 \dot{\mathcal{P}}^1 = \{Q, \bar{\mathcal{C}}_1 \dot{N}^1\}$  have been suppressed by replacing  $\chi^1$  with  $\chi^1 + \dot{N}^1$ .

Now we perform path integration over the ghosts, anti-ghosts and Lagrangian multipliers  $\bar{\mathcal{C}}_1$ ,  $\mathcal{P}^1$ ,  $\bar{\mathcal{P}}_1$ ,  $\mathcal{C}^1$ ,  $B_1$  and  $N^1$ , by using the equations of motion. This leads to the effective Lagrangian of the form

$$\begin{aligned} L_{eff} = & p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} + p_\eta \dot{\eta} + B \dot{N} + \bar{\mathcal{P}} \dot{\mathcal{C}} + \bar{\mathcal{C}} \dot{\mathcal{P}} - \frac{(p_r + p_\eta)^2}{2m} - \frac{p_\theta^2}{2m(r - \eta)^2} - \frac{p_\phi^2}{2m(b + (r - \eta) \sin \theta)^2} \\ & + (p_r + p_\eta) N + B p_r + \bar{\mathcal{P}} \mathcal{P} \end{aligned} \quad (3.8)$$

with the redefinitions:  $N = N^2$ ,  $B = B_2$ ,  $\bar{\mathcal{C}} = \bar{\mathcal{C}}_2$ ,  $\mathcal{C} = \mathcal{C}^2$ ,  $\bar{\mathcal{P}} = \bar{\mathcal{P}}_2$ ,  $\mathcal{P} = \mathcal{P}_2$ .

After performing the routine variation procedure and identifying  $N = -B - \dot{\eta}$  we arrive at the effective Lagrangian of the form

$$L_{eff} = L_0 + L_{WZ} + L_{gh}, \quad (3.9)$$

where  $L_0$  is given by (2.1) and  $L_{WZ}$  and  $L_{gh}$  are given by

$$\begin{aligned} L_{WZ} = & \frac{1}{2} m \dot{\eta}(\dot{\eta} - 2\dot{r}) + \frac{1}{2} m \eta(\eta - 2r)\dot{\theta}^2 + \frac{1}{2} m \eta \sin \theta (\eta \sin \theta - 2b - 2r \sin \theta) \dot{\phi}^2 \\ L_{gh} = & \dot{B} \dot{\eta} + \dot{\bar{\mathcal{C}}} \dot{\mathcal{C}}. \end{aligned} \quad (3.10)$$

The effective Lagrangian  $L_{eff}$  is now invariant under the BRST-transformation

$$\begin{aligned} \delta_B r = & \lambda \mathcal{C}, \quad \delta_B \theta = 0, \quad \delta_B \phi = 0, \quad \delta_B \eta = \lambda \mathcal{C}, \\ \delta_B \bar{\mathcal{C}} = & -\lambda B, \quad \delta_B \mathcal{C} = 0, \quad \delta_B B = 0. \end{aligned} \quad (3.11)$$

#### IV. ENERGY SPECTRUM AND PARTICLE DYNAMICS

Now, in order to investigate the energy spectrum of the free particle system on a torus, we impose the first-class constraints (2.14) on the first-class Hamiltonian  $\tilde{H}_0$  in (2.15) to yield

$$\tilde{H}_0 = \frac{p_\theta^2}{2ma^2} + \frac{p_\phi^2}{2m(b + a \sin \theta)^2}. \quad (4.1)$$

Since the free particle of interest is constrained to reside on the torus, we should include the geometrical effects of the target manifold of the torus whose spatial two-metric is given by

$$ds^2 = a^2 d\theta^2 + (b + a \sin \theta)^2 d\phi^2. \quad (4.2)$$

The natural choice of zweibein frame is then

$$e_\theta = \frac{1}{a} \frac{\partial}{\partial \theta}, \quad e_\phi = \frac{1}{b + a \sin \theta} \frac{\partial}{\partial \phi}, \quad (4.3)$$

to, together with the commutator relations (2.13), yield the Hamiltonian operator

$$\tilde{H}_0 = -\frac{\hbar^2}{2m} \left[ \frac{1}{a^2(b + a \sin \theta)} \frac{\partial}{\partial \theta} \left( (b + a \sin \theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{(b + a \sin \theta)^2} \frac{\partial^2}{\partial \phi^2} \right]. \quad (4.4)$$

Note that in the  $b \rightarrow 0$  limit, the Hamiltonian operator (4.4) on the torus reduces to that on a two-sphere.<sup>2</sup>

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<sup>2</sup> The two-sphere Laplacian is given by  $\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$  in spherical coordinates and it can be rewritten as  $\partial_i \partial_i - \frac{2x_i}{x_k x_k} \partial_i - \frac{x_i x_j}{x_k x_k} \partial_i \partial_j$  in Cartesian coordinates [20].

Next, we consider an eigenvalue equation of the form

$$\tilde{H}_0\psi(\theta, \phi) = E\psi(\theta, \phi). \quad (4.5)$$

Firstly, for a given angle  $\theta$ , we can have a reduced Schrödinger equation

$$-\frac{\hbar^2}{2m(b+a\sin\theta)^2}\frac{d^2\psi}{d\phi^2} = E\psi \quad (4.6)$$

to yield the eigenfunctions

$$\psi_l(\phi) = \frac{e^{il\phi}}{(2\pi)^{1/2}}, \quad (4.7)$$

with  $l = 0, \pm 1, \pm 2, \dots$  and the energy spectrum of the particle zero modes

$$E_l(\theta) = \frac{\hbar^2 l^2}{2m(b+a\sin\theta)^2}. \quad (4.8)$$

Note that in the limit of  $b \gg a$ , we can obtain the  $\theta$ -independent form

$$E_l = \frac{\hbar^2 l^2}{2I_b}, \quad (4.9)$$

which describes the particle motion, with quantum number  $l$ , rotating on an axially circular orbit of radius  $b$  along the  $\phi$ -direction. Here, the moment of inertia of the particle is given by

$$I_b = mb^2, \quad (4.10)$$

and  $l$  is the angular momentum quantum number of the corresponding operator  $J_\phi$ . Note that the quantum operator  $J_\phi$  is defined on the two-dimensional  $x_1$ - $x_2$  plane to yield the quantum number  $l^2$  of  $\vec{J}_\phi^2$ . Moreover, the angular momentum operator  $\vec{J}_\phi^2$  produces the quantum number, instead of  $l(l+1)$ ,  $l^2$  which is a characteristic of two-dimensional rigid rotator [21].

Secondly, for a given angle  $\phi$ , we can find a Schrödinger equation of the form

$$-\frac{\hbar^2}{2ma^2(b+a\sin\theta)}\frac{d}{d\theta}\left((b+a\sin\theta)\frac{d\psi}{d\theta}\right) = E\psi. \quad (4.11)$$

Setting

$$\psi_n(\theta) = e^{in\theta}\Theta(\theta), \quad (4.12)$$

we can decompose the second-order Schrödinger equation (4.11) into two ordinary differential equations

$$\frac{d^2\Theta}{d\theta^2} + \frac{a\cos\theta}{b+a\sin\theta}\frac{d\Theta}{d\theta} + \left(\frac{2ma^2E}{\hbar^2} - n^2\right)\Theta = 0, \quad (4.13)$$

and

$$\frac{d\Theta}{d\theta} + \frac{a\cos\theta}{2(b+a\sin\theta)}\Theta = 0, \quad (4.14)$$

from which we can obtain the eigenfunctions with  $n = 0, \pm 1, \pm 2, \dots$

$$\psi_n(\theta) = \frac{(b^2-a^2)^{1/4}}{(2\pi)^{1/2}}\frac{e^{in\theta}}{(b+a\sin\theta)^{1/2}}, \quad (4.15)$$

and the energy spectrum of the particle zero modes

$$E_n(\theta) = \frac{\hbar^2}{2m}\left[\frac{n^2}{a^2} - \frac{\cos^2\theta}{4(b+a\sin\theta)^2} - \frac{\sin\theta}{2a(b+a\sin\theta)}\right]. \quad (4.16)$$

In the limit of  $b \gg a$ , we can obtain the eigenfunctions of the particle zero modes

$$\psi_n(\theta) = \frac{e^{in\theta}}{(2\pi)^{1/2}}, \quad (4.17)$$

and the corresponding energy spectrum

$$E_n = \frac{\hbar^2 n^2}{2I_a}, \quad (4.18)$$

where the moment of inertia  $I_a$  of the particle is given by

$$I_a = ma^2, \quad (4.19)$$

and  $n$  is the angular momentum quantum number of the corresponding operator  $J_\theta$ . Note that, on the two-dimensional cross sectional constant- $\phi$  plane, the quantum operator  $J_\theta$  is well defined to produce the quantum number  $n^2$  of  $\vec{J}_\theta^2$ , as mentioned in (4.9). In fact, the energy spectrum in the limit of  $b \gg a$  denotes the particle motion, with angular momentum quantum number  $n$ , rotating on a circular orbit of radius  $a$  along the  $\theta$ -direction.

Thirdly, for a general case of (4.5), we again set

$$\psi_{nl}(\theta, \phi) = e^{i(n\theta+l\phi)}\Theta(\theta), \quad (4.20)$$

to yield the first-order differential equation (4.14) and a second-order differential equation

$$\frac{d^2\Theta}{d\theta^2} + \frac{a \cos \theta}{b + a \sin \theta} \frac{d\Theta}{d\theta} + \left[ \frac{2ma^2 E}{\hbar^2} - \frac{a^2 l^2}{(b + a \sin \theta)^2} - n^2 \right] \Theta = 0. \quad (4.21)$$

After some algebra, we can find the eigenfunctions with the quantum numbers  $n = 0, \pm 1, \pm 2, \dots$  and  $l = 0, \pm 1, \pm 2, \dots$

$$\psi_{nl}(\theta, \phi) = \frac{(b^2 - a^2)^{1/4}}{2\pi} \frac{e^{i(n\theta+l\phi)}}{(b + a \sin \theta)^{1/2}}, \quad (4.22)$$

and the energy spectrum of the particle zero modes

$$E_{nl}(\theta) = \frac{\hbar^2}{2m} \left[ \frac{n^2}{a^2} + \frac{4l^2 - \cos^2 \theta}{4(b + a \sin \theta)^2} - \frac{\sin \theta}{2a(b + a \sin \theta)} \right]. \quad (4.23)$$

Note that the energy spectrum (4.23) is the most general solution to the Schrödinger equation (4.5) for the quantum Hamiltonian operator (4.4) and the corresponding eigenfunctions (4.22) satisfy the following orthogonality condition

$$\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \psi_{nl}(\theta, \phi) \psi_{n'l'}(\theta, \phi) = \delta_{nn'} \delta_{ll'}. \quad (4.24)$$

In order to investigate the particle dynamics associated with the energy spectrum structure in terms of the toric geometrical parameters  $a$  and  $b$ , we consider the simple case of  $b \gg a$  to arrive at the eigenfunctions

$$\psi_{nl}(\theta, \phi) = \frac{e^{i(n\theta+l\phi)}}{2\pi}, \quad (4.25)$$

and the corresponding  $\theta$ -independent energy spectrum of the particle zero modes

$$E_{nl} = \frac{\hbar^2}{2} \left( \frac{n^2}{I_a} + \frac{l^2}{I_b} \right), \quad (4.26)$$

where  $I_a$  and  $I_b$  are moments of inertia defined in (4.19) and (4.10), respectively. Note that in the case of  $(n, l) = (0, l)$  the energy spectrum (4.26) is reduced to that of (4.9), while in the case of  $(n, l) = (n, 0)$  to that of (4.18). In the case of the angular momentum quantum numbers  $(n, l)$  associated with the operators  $J_\theta$  and  $J_\phi$  discussed above, the energy spectrum (4.26) describes the particle motion, rotating on a helix orbit of radius  $a$  along the  $\theta$ -direction and pointing toward the  $\phi$ -direction of the torus of axial radius  $b$ .

## V. SYMPLECTIC STRUCTURES

In this section, we show that the Dirac brackets obtained in the previous section are in full agreement with those in the symplectic approach [18] to our second-class system. We start with considering the symplectic analogue of the conventional Dirac approach. The master Lagrangian given by  $L_0$  in (2.1) is of the form

$$L^{(0)} = a_\alpha^{(0)} \dot{\xi}^{(0)\alpha} - V^{(0)} \quad (5.1)$$

where

$$\xi^{(0)\alpha} = (\theta, p_\theta, \phi, p_\phi, p_r), \quad a_\alpha^{(0)} = (p_\theta, 0, p_\phi, 0, 0), \quad (5.2)$$

and  $V^{(0)}$  is given by  $H_0$  in (2.7). The Euler-Lagrange equations then read

$$f_{\alpha\beta}^{(0)} \dot{\xi}^{(0)\beta} = K_\alpha^{(0)}, \quad (5.3)$$

where  $f_{\alpha\beta}^{(0)}$  is the (pre-)symplectic form

$$f_{\alpha\beta}^{(0)} = \frac{\partial a_\beta^{(0)}}{\partial \xi^{(0)\alpha}} - \frac{\partial a_\alpha^{(0)}}{\partial \xi^{(0)\beta}}, \quad (5.4)$$

and  $K_\alpha^{(0)}$  is given by

$$K_\alpha^{(0)} = \frac{\partial V^{(0)}}{\partial \xi^{(0)\alpha}} \quad (5.5)$$

Explicitly we obtain

$$f^{(0)} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K^{(0)} = \begin{pmatrix} -\frac{p_\phi^2 r \cos \theta}{m(b+r \sin \theta)^3} \\ \frac{p_\theta}{mr^2} \\ 0 \\ \frac{p_\phi}{m(b+r \sin \theta)^2} \\ \frac{p_r}{m} \end{pmatrix}. \quad (5.6)$$

It is evident that since  $\det f^{(0)} = 0$ , the matrix  $f^{(0)}$  is not invertible. In fact the rank of this matrix is 4, so that there exists infinity of zero-generation (left) zero mode  $\nu_\alpha^{(0)}$  as follows

$$\nu^{(0)T} = (0, 0, 0, 0, 1), \quad (5.7)$$

where the superscript  $T$  stands for transpose. Correspondingly, we have an infinity of zero-generation constraint

$$\nu_\alpha^{(0)T} \frac{\partial V^{(0)}}{\partial \xi^{(0)\alpha}} = 0 \quad (5.8)$$

which leads to the constraint  $\Omega_2 = p_r$  in (2.9). Our new set of first-generation dynamical variables are then given by

$$\xi^{(1)\alpha} = (\theta, p_\theta, \phi, p_\phi, p_r, \rho), \quad (5.9)$$

and the first-generation Lagrangian takes the form

$$L^{(1)} = a_\alpha^{(1)} \dot{\xi}^{(1)\alpha} - V^{(1)} \quad (5.10)$$

where

$$a_\alpha^{(1)} = (p_\theta, 0, p_\phi, 0, 0, \Omega_2), \quad (5.11)$$

and

$$V^{(1)} = \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2m(b+r \sin \theta)^2}. \quad (5.12)$$

The Euler-Lagrange equations now takes the form

$$f_{\alpha\beta}^{(1)} \dot{\xi}^{(1)\beta} = K_\alpha^{(1)}, \quad (5.13)$$

where the first-generation symplectic form  $f_{\alpha\beta}^{(1)}$  is given by

$$f_{\alpha\beta}^{(1)} = \frac{\partial a_\beta^{(1)}}{\partial \xi^{(1)\alpha}} - \frac{\partial a_\alpha^{(1)}}{\partial \xi^{(1)\beta}}, \quad (5.14)$$

and  $K_\alpha^{(1)}$  is given by

$$K_\alpha^{(1)} = \frac{\partial V^{(1)}}{\partial \xi^{(1)\alpha}}. \quad (5.15)$$

We then obtain explicitly

$$f^{(1)} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad K^{(1)} = \begin{pmatrix} -\frac{p_\phi^2 r \cos \theta}{m(b+r \sin \theta)^3} \\ \frac{p_\theta}{mr^2} \\ 0 \\ \frac{p_\phi}{m(b+r \sin \theta)^2} \\ 0 \\ 0 \end{pmatrix}. \quad (5.16)$$

Moreover, the inverse  $f_{-1}^{(1)}$  of the matrix  $f^{(1)}$  is given by

$$f_{-1}^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \quad (5.17)$$

Now, let  $F$  and  $G$  be functions of the dynamical variables  $\xi^{(1)\alpha}$ . We can then define generalized symplectic structures as

$$\{F, G\}^* = \frac{\partial F}{\partial \xi^{(1)\alpha}} f_{-1}^{(1)\alpha\beta} \frac{\partial G}{\partial \xi^{(1)\beta}}. \quad (5.18)$$

In particular, we have the following symplectic structures

$$\{\xi^{(1)\alpha}, \xi^{(1)\beta}\}^* = f_{-1}^{(1)\alpha\beta}, \quad (5.19)$$

to yield

$$\{\theta, p_\theta\}^* = 1, \quad \{\phi, p_\phi\}^* = 1, \quad \{\rho, p_r\}^* = 0. \quad (5.20)$$

Note that, with the identification  $\rho = r$ , the symplectic brackets in (5.20) reproduce the Dirac brackets in (2.12). Moreover, together with the matrices (5.16), the Euler-Lagrange equations (5.13) reconstruct the canonical momenta (2.8), to yield

$$\dot{p}_\theta = \frac{p_\phi^2 r \cos \theta}{m(b+r \sin \theta)^3} = mr(b+r \sin \theta) \cos \theta \dot{\phi}^2, \quad \dot{p}_\phi = 0 \quad (5.21)$$

attainable also from the variations of the Lagrangian (2.1) with respect to the coordinates  $\theta$  and  $\phi$ , and the consistency conditions of the constraints associated with the coordinate  $r = \rho$

$$\dot{\Omega}_1 = 0, \quad \dot{\Omega}_2 = 0. \quad (5.22)$$

## VI. CONCLUSION

In conclusion, we have constructed a first-class Hamiltonian by exploiting the Stückelberg coordinates associated with the geometrical constraints imposed on the torus. Subsequently, in the BFV scheme we have obtained the BRST-invariant gauge fixed Lagrangian including the ghosts and anti-ghosts, and the BRST transformation rules under which the effective Lagrangian is invariant. We have also discussed the particle dynamics by constructing the spectrum of the free particle system on the torus. The symplectic structures involved in the system on the torus were also investigated to yield the consistency with the Dirac analysis for the second-class constraints. It would be interesting to study a field theoretic extension of this work with toric geometric constraints.

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